The Emergence of Anticommuting Coordinates and the Dirac-Ramond-Kostant operators

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1   Introduction

Gunnar Nordström died just a few years before the advent of quantum mechanics. It is is very interesting, but of course futile to speculate how he would have contributed to the new subject. His early attempts to increase the number of space dimensions [1] could very well have been very important for him when the spin degrees of freedom were understood. The birth of quantum mechanics in the 1920's with the papers of Heisenberg [2] and Schrödinger [3] and others really made a quantum jump in our understanding of the microcosm. The formulation was revolutionary but treated space and time somewhat conservatively, since it was non-relativistic and the newly discovered spin degrees of freedom were just added on as multiplicative wave functions not connected to the 3-dimensional space. In the hectic year of 1926 Klein [4], Gordon [5] and Schrödinger [3] discussed the relativistic version and the Klein- Gordon equation was suggested.

$$((-i\hbar\frac{\partial}{\partial x^\mu})^2 + m^2)\Psi = 0. \tag{1}$$

Schrödinger originally discarded this equation since it does not give the correct spectrum for the hydrogen atom. In order to remedy this problem, Darwin aand Pauli [6] added spin terms and succeeded to give a good but not perfect match of the spectrum.

Only Dirac seemed to worry about this problem. He talked to Bohr about it when he visited Copenhagen, but Bohr assured him that the problem would be solved within the existing formalism. Dirac had other problems with the Klein-Gordon equation. He found that there is non-positive probability following from the equation. Consider the conserved current

$$J^0 = \quad -i(\Psi^*\frac{\partial}{\partial t}\Psi - \frac{\partial}{\partial t}\Psi^* \ \Psi) \tag{2}$$

$$J^i = \ -i(\Psi^*\frac{\partial}{\partial x^i}\Psi - \frac{\partial}{\partial x^i}\Psi^* \ \Psi). \tag{3}$$

Dirac wanted to have a probability $|\Psi|^2$. Then one sacred evening in front of the fireplace at St John's College he found the answer to his problems. The

equation must be linear in $\frac{\partial}{\partial t}$ and he was led to the equation that would once be on his tombstone in Westminster Abbey [7].

$$i\hbar \frac{\partial \Psi}{\partial t} = (-i\hbar \overrightarrow{\alpha} \cdot \overrightarrow{\partial} + \alpha_4 m)\Psi \tag{4}$$

or in more modern form

$$(i\gamma_\mu \partial^\mu - m)\Psi = 0. \tag{5}$$

For this equation Dirac found a positive probability through the conserved current

$$J^0 = \bar{\Psi}\gamma^0 \Psi = |\Psi|^2 \tag{6}$$
$$J^i = -\bar{\Psi}\gamma^i \Psi. \tag{7}$$

There was a phenomenal success for the equation. It really gave the correct spectrum for the hydrogen atom taking spin into account. However, as it happens for some of the most gifted scientists, the result was correct but the starting point not fully correct. With the understanding of the positron solution one understood that the Dirac current should be multiplied by the charge and should be interpreted as a current for the charge leading to positive or negative charge densities depending on the sign of the charges present. Similarly the Klein-Gordon equation was understood to describe scalar particles and again the current should be understood as a charge current. It is fair to say that the Dirac equation was the starting point for the Standard Model. The nonconservation of particles led to quantum field theory and eventually it was realized that such theories need non-abelian gauge fields for $d = 4$.

There was one observation in Dirac's paper that never caught on at the time. It is natural since the equation worked so well so one did not need to theorize over the origin of the equation. Dirac made a point that the γ's are dynamical and on an equal footing to x^μ and p^μ. That point was dormant for more than forty years until Ramond utilized this fact when he discovered superstring theory [8].

There were other issues that nobody really raised. The algebra between $\gamma \cdot p$ and p^2 is a graded algebra and this was in fact the first graded algebra in physics, but its mathematical properties were not really exploited. Neither did one discuss the obvious graded algebra between Dirac fermions and Klein-Gordon bosons, which is the basis for supersymmetry. The use of Grassmann variables was not understood until fermions were used in Lagrangian formulations of quantum field theories. All these properties came up in the 1970's when we tried to find new models. In Dirac's time he saw no need for new models. The equation worked so beautifully.

One can speculate what would have happened if Dirac had stayed in Bristol and had been shielded from the events of 1925 and 1926. Perhaps he would have discovered his formalism from 1949 first [9]. In that paper Dirac showed

that any direction inside the light-cone including the limiting light-cone itself can be used as the evolution parameter for the relativistic problem. Specifically one can use $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^{d-1})$. From the mass-shell relation

$$p^2 + m^2 = -2p^+p^- + p^{i^2} + m^2 = 0, \qquad (8)$$

one can solve for

$$p^- = \frac{1}{2p^+}(p^{i^2} + m^2). \qquad (9)$$

This equation is linear in the light-cone energy p^- and the corresponding wave equation reads

$$-i\hbar\partial^-\Psi = (-\hbar^2\frac{1}{2p^+}\partial^{i^2} + \frac{m^2}{2p^+})\Psi. \qquad (10)$$

The equation is identical for spin-0 and spin-1/2 particles. They differ in their Grassmann properties of the Ψ only. However, on our planet the development went the other way.

2 Dual Resonance Models

String theory was born in 1968 with the celebrated paper by Veneziano [10]. He constructed a four-point amplitude with magical properties. In 1969 Koba and Nielsen [11] found the N-point generalization of this amplitude which subsequently was factorized by Fubini and Veneziano [12] and by Nambu [13]. The important constructs are

$$Q^\mu(z) = x^\mu - ip^\mu lnz - i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(a^\dagger{}_n{}^\mu z^n - a_n{}^\mu z^{-\mu}). \qquad (11)$$

$$P^\mu = iz\frac{dQ^\mu}{dz}, \qquad (12)$$

which generalize the usual phase space of a point-particle. Since these amplitudes describe open strings in modern terms, the world sheet variable $z = exp(i\tau)$. In Koba-Nielsen's work it was just an internal parameter. The amplitudes were very interesting but they lacked fermions.

On January 4, 1971 Pierre Ramond [8] submitted a paper which was the starting point for the Superstring Theory. In this paper he used the insight that Dirac had back in 1926 and put it into practise into dual resonance models, the name that the subject had at the time. He realized that the x and the p in the preceeding formulae are zero modes and he devised an averaging procedure which is like taking the zero fourier mode, where

$$x^\mu = \quad < Q^\mu(z) >, p^\mu = \quad < P^\mu(z) > . \tag{13}$$

and a correspondence principle for how to go from the usual field theory equations (Klein-Gordon) to the dual resonance models equations

$$p^2 + m^2 = < P^\mu >< P_\mu > + m^2 \Rightarrow \tag{14}$$

$$(< P^\mu P_\mu > + m^2) \, |phys> = 0 \qquad , \tag{15}$$

or

$$(L_0 - m^2)|phys> = 0. \tag{16}$$

Similarly the Virasoro operators [14] are given by

$$L_n = < z^n \, P^2(z) > . \tag{17}$$

In this formalism it is natural to add in fermions. Remember that Dirac said that the γ's should be regarded as dynamical variables. We can now use the correspondence principle and introduce a $\Gamma^\mu(z)$ such that

$$\gamma^\mu = < \Gamma^\mu(z) > \tag{18}$$

with the natural anticommutation relation

$$\{\Gamma^\mu(z), \Gamma^\nu(z')\} = 2\eta^{\mu\nu}\delta(z - z'). \tag{19}$$

A solution to this is

$$\Gamma^\mu(z) = \gamma^\mu + i\sqrt{2}\gamma^5 \sum_{n=1}^{\infty} (b^\dagger{}_n{}^\mu z^n + b_n{}^\mu z^{-n}). \tag{20}$$

It is now rather natural to try to generalize the Dirac equation in the same way the Klein-Gordon equation led to the Virasoro constraints.

$$\gamma \cdot p - m = < \Gamma_\mu(z) >< P^\mu(z) > - m = 0 \Rightarrow \tag{21}$$

$$[< \Gamma_\mu(z)P^\mu(z) > - m] \, |phys> = 0. \tag{22}$$

Square the two expressions.

$$< \Gamma \cdot P >< \Gamma \cdot P > - m^2 \ = 0 \Rightarrow \tag{23}$$

$$[< P^2(z) > - i/4 < \Gamma(z)z\tfrac{d}{dz}\Gamma(z) > - m^2] \, |phys> = 0. \tag{24}$$

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This is the new mass shell condition $L_0 - m^2 = 0$ and the fourier modes are the new Virasoro operators

$$L_n = < z^n P^2(z) > -i/4 < z^{n+1}\Gamma(z)z\frac{d}{dz}\Gamma(z) > . \tag{25}$$

Similarly the Dirac equation led to the constraint operators

$$F_n = < z^n \Gamma_\mu(z)P^\mu(z) >, \tag{26}$$

which together with the new L_n's give the superVirasoro algebra and hence opened the way to construct scattering amplitudes involving propagating fermions.

This was the first example of a 2-dimensional superconformal algebra extending a 2-dimensional supersymmetry and it led eventually to 4-dimensional supersymmetry [15]. The rest is history.

3   Anticommuting coordinates

It is clear from the discussion above that the γ-matrices and their extensions should be interpreted as being dynamical coordinates carrying degrees of freedom. If one considers the classical action for a scalar particle and the one for a spinning one, the relation between the usual coordinates and the fermionic ones become clear.

The usual action for a spinless particle

$$S = m \int d\tau \sqrt{-\dot{x}^2} \tag{27}$$

can be rewritten as

$$S = \frac{1}{2} \int d\tau (e\dot{x}^2 - \frac{1}{e}m^2). \tag{28}$$

The function e is an einbein. The latter form can be used for the massless case. This expression lends itself to a generalization to include Grassmann odd coordinates ψ^μ [16].

$$S = \frac{1}{2} \int d\tau (e\dot{x}^2 + \psi \cdot \dot{\psi} + \lambda\dot{x}\psi). \tag{29}$$

In here λ is a superpartner to the einbein e. There is a (necessary) difference between the bosonic coordinate x and the fermionic one ψ in that the latter occurs with just one derivative in the action. This leads to a constrained phase space with no momentum conjugate to ψ. (We can also write it as $d/2$ coordinates and $d/2$ momenta.)

In the quantum case the usual representation is

$$\psi^\mu \rightarrow \gamma^\mu. \tag{30}$$

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We can also introduce a more coordinate-like representation by introducing (in the 4-dimensional case as an example) two Grassmann odd coordinates θ^1 and θ^2 [17] and write

$$\psi^0 + \psi^3 \quad = \quad \sqrt{2}\,\theta^1, \tag{31}$$

$$\psi^1 + i\psi^2 \quad = \quad \sqrt{2}\,\theta^2, \tag{32}$$

$$\psi^0 - \psi^3 \quad = \quad \sqrt{2}\,\frac{d}{d\theta^1}, \tag{33}$$

$$\psi^1 - i\psi^2 \quad = \quad \sqrt{2}\,\frac{d}{d\theta^2}. \tag{34}$$

The corresponding field will then be a function of x^μ, θ^1, θ^2, ie have four components like a spinor.

Another type of fermionic coordinate is the spinorial one θ^a, which was pioneered by Montonen [18] in string theory and then taken over to 4 dimensions by Salam and Strathdee [19]. We get that if we instead of the vectorial fermionic coordinate in (29) use a spinorial one and change the action to the action for the superparticle [20].

4   What the Physicists did not do

Consider again the Dirac operator $\slashed{P} = \gamma_\mu P^\mu$. We can regard the P^μ's as generators for the translation algebra. The Dirac operator satisfies the algebra

$$\{\slashed{P},\ \slashed{P}\} = P^2. \tag{35}$$

If we let P^2 act on a state

$$P^2\Psi = 0, \tag{36}$$

the result is nontrivial since the translation algebra is non-compact. What happens if we generalize this procedure to a Lie algebra?

$$[T^a,\ T^b] = if^{abc}T^c, \quad a = 1, 2, \ldots D. \tag{37}$$

Define the Kostant-Dirac operator [21] as

$$\slashed{K} = \gamma^a T^a - \frac{i}{2 \cdot 3!}\gamma^{abc}f^{abc} \tag{38}$$

with $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$ and γ^{abc}, the three times antisymmetrized product of three γ's. Then

$$\{\slashed{K},\ \slashed{K}\} = 2C_2 + \frac{D}{12}C_2{}^{adjt}, \tag{39}$$

where C_2 is the Casimir operator of the representation of T^a and $C_2{}^{adjt}$ is the Casimir operator of the adjoint representation. If we consider the Kostant-Dirac equation

$$\not{K}\Psi = 0, \tag{40}$$

one finds that it has only trivial solutions for a compact algebra.

Consider instead Lie algebra cosets. Let

g be generated by T^a $a = 1 \ldots D$,
h be generated by T^i $i = 1 \ldots d$,
g/h be generated by T^m $m = 1 \ldots D - d$.

Consider so the Kostant operator for the coset g/h.

$$\not{K} = \gamma^m T^m - \frac{i}{2 \cdot 3!} \gamma^{mnp} f^{mnp}. \tag{41}$$

Now the square of the operator is

$$\not{K}^2 = [C_2(g) + \frac{D}{24} C_2{}^{adjt}(g)] - [C_2(h) + \frac{D}{24} C_2{}^{adjt}(h)], \tag{42}$$

where h is generated by $L^i = T^i + S^i$ and $S^i = -\frac{i}{4} \gamma^{mn} f^{imn}$. This formula is remarkable in that the spin operators for the subalgebra use the structure constants of the full algebra.

The full generator L^i of the subalgebra commutes with the Kostant operator $\not{K}$ and hence the solutions to the the Kostant equation

$$\not{K}\Psi = 0 \tag{43}$$

have solutions assembled in full multiplets of h.

Take as an example g to be the Poincaré algebra and h to be the Lorentz one. The Lorentz algebra is then generated by

$$M^{\mu\nu} = T^{\mu\nu} + S^{\mu\nu}, \tag{44}$$

where

$$S^{\mu\nu} = \frac{i}{4} \gamma^{\rho\sigma} f^{(\mu\nu)\rho\sigma} \tag{45}$$

$$= -\frac{i}{2} \gamma^{\mu\nu}, \tag{46}$$

just as Dirac once told us.

If g and h have the same rank the solution to Kostant's equation can be written as [22]

$$\Psi = V_\lambda \otimes S^+ - V_\lambda \otimes S^- \,, \tag{47}$$

where S are the two spinor representation of $SO(D-d)$ and V_λ is the representation of g of highest weight λ. The solution is an infinite sequence of r-tuplets with r the Euler number if g/h. In some cases, the r-tuplets, or Euler multiplets, have the same number of bosons and fermions [23], but no apparent supersymmetry, which still make them very interesting from a quantum mechanical point of view. Indeed, it has been shown that

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_c^r (-1)^w \, U_{c\bullet\lambda} \,, \tag{48}$$

where the sum is over the r elements of the Weyl group of **g** that are not in **h**'s, and $U_{c\bullet\lambda}$ denote representations of **h**, and the $\bullet$ denotes a specific construction which has been defined in [22].

4.1   Supersymmetric Euler Multiplets

It is well-known that the degrees of freedom of supersymmetric theories can be labelled in terms of the Wigner little group of the associated Poincaré algebra. Since theories with gravity cannot sustain a finite number of *massless* degrees of freedom with spin higher than two, the cosets cannot exceed sixteen dimensions, which yield the two spinor representations of $SO(16)$, with 256 degrees of freedom.

We have found [24] only a few cosets, with $D - d = 16, 8, 4$, for which the basic Euler multiplets have the right quantum numbers to represent massless particles of relativistic theories:

▷ **16-dimensional Cosets**

The 256 states of the associated Clifford are generated by the two spinor irreps of $SO(16)$, yielding three possible theories:

- $SU(9) \supset SU(8) \times U(1)$ with lowest Euler multiplet

  $$\mathbf{1}_2 \oplus \mathbf{8}_{3/2} \oplus \mathbf{28}_1 \oplus \mathbf{56}_{1/2} \oplus \mathbf{70}_0 \oplus \overline{\mathbf{56}}_{-1/2} \oplus \overline{\mathbf{28}}_{-1} \oplus \overline{\mathbf{8}}_{-3/2} \oplus \mathbf{1}_{-2}$$

  Interpreting the $U(1)$ as the helicity little group in four dimensions, these can be thought of as the massless spectrum of either type IIB string theory, or of $N = 8$ supergravity.

- $SO(10) \supset SO(8) \times SO(2)$ with lowest Euler multiplet

  $$\mathbf{1}_2 \oplus \mathbf{8}_{3/2} \oplus \mathbf{28}_1 \oplus \mathbf{56}_{1/2} \oplus (\mathbf{35}_0 \oplus \mathbf{35}_0) \oplus \mathbf{56}_{-1/2} \oplus \mathbf{28}_{-1} \oplus \mathbf{8}_{-3/2} \oplus \mathbf{1}_{-2}$$

Viewing $SO(8)$ as the little group in ten dimensions, these may represent the massless spectrum of IIB superstring in ten dimensions, but only after using $SO(8)$ triality. Alternatively, with $U(1)$ as the helicity, it could just be the massless particle content of the type IIB superstring theory, or of $N = 8$ supergravity in four dimensions.

- $SU(6) \supset SO(6) \times SO(3) \times SO(2)$ with lowest Euler multiplet

$$(\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{4}, \mathbf{2})_{3/2} \oplus (\mathbf{10}, \mathbf{1})_1 \oplus (\mathbf{6}, \mathbf{3})_1 \oplus (\mathbf{4}, \mathbf{4})_{1/2} \oplus (\mathbf{20}, \mathbf{2})_{1/2}$$

$$\oplus (\mathbf{20'}, \mathbf{1})_0 \oplus (\mathbf{15}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{5})_0$$

$$\oplus (\mathbf{4}, \mathbf{4})_{-1/2} \oplus (\overline{\mathbf{20}}, \mathbf{2})_{-1/2} \oplus (\overline{\mathbf{10}}, \mathbf{1})_{-1} \oplus (\mathbf{6}, \mathbf{3})_{-1} \oplus (\overline{\mathbf{4}}, \mathbf{2})_{-3/2} \oplus (\mathbf{1}, \mathbf{1})_{-2} \ .$$

This is the massless spectrum of $N = 8$ supergravity, or of type IIB superstring in four dimensions, or of massless theories in five and eight dimensions.

- $F_4 \supset SO(9)$ The lowest Euler multiplet

$$\mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$$

describes the massless spectrum of $N = 1$ supergravity in 11 dimensions, the local limit of M-theory, by identifying $SO(9)$ as the massless little group.

▷ **8-dimensional Cosets**

The Clifford algebra is realized on the 16 states that span the two spinor irreps of $SO(8)$, leading to the two massless interpretations

- $SU(5) \supset SU(4) \times U(1)$ which yields the lowest Euler multiplet

$$\mathbf{1}_1 \oplus \mathbf{4}_{1/2} \oplus \mathbf{6}_0 \oplus \overline{\mathbf{4}}_{-1/2} \oplus \mathbf{1}_{-1}$$

With $U(1)$ as the helicity little group, this particle content is the same as the massless spectrum of $N = 4$ Yang-Mills in four dimensions.

It could also describe a massless theory in eight dimensions with one conjugate spinor, and eight scalars, since $SU(4) \sim SO(6)$.

- $SO(6) \supset SO(4) \times SO(2)$ with lowest Euler multiplet

$$(\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{2}, \mathbf{2})_{1/2} \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{3}, \mathbf{1})_0 \oplus (\mathbf{2}, \mathbf{2})_{-1/2} \oplus (\mathbf{1}, \mathbf{1})_{-1}$$

This particle content is the same as the $N = 4$ Yang-Mills in four dimensions.

▷ 4-**dimensional Cosets**

The Clifford algebra is realized on 4 states of the two spinor irreps of $SO(4)$. They appear as the lowest Euler multiplet in the following decomposition:

- $SU(3) \supset SU(2) \times U(1)$. The lowest Euler multiplet

$$\mathbf{1}_{1/2} \oplus \mathbf{2}_0 \oplus \mathbf{1}_{-1/2}$$

  can be interpreted as a relativistic theory in 4 dimensions, since the $U(1)$ charges are half integers and integers, and it can describe the massless $N = 1$ Wess-Zumino supermultiplet in four dimensions, as well as a massless supermultiplet in five dimensions.

So far, we have limited our discussion to massless degrees of freedom, but we should note for completeness that possible relativistic theories with massive degrees of freedom appear in the cosets $Sp(2P + 2) \supset Sp(2P) \times Sp(2)$ with the same content as massive $N = P$ supersymmetry in four dimensions. The massive $N = 4$ supermultiplet in four dimensions is also described by the coset $SO(8) \supset SO(4) \times SO(4)$, and the massive $N = 2$ supermultiplet in four dimensions can be described in terms of $G_2/SU(2) \times SU(2)$.

In this section, we have shown that the degrees of freedom of some supersymmetric theories are nothing but solutions of the Kostant-Dirac equation associated with specific cosets. Furthermore, not all supersymmetric theories appear in this list, only the local limit of M-theory, of type IIB superstring theories (not type IIA, type I, nor heterotic superstrings), and certain local field theories, $N = 4$ Yang-Mills, and $N = 1$ Wess-Zumino multiplets. All the massless cosets are both hermitian and symmetric, except for the Wess-Zumino multiplet and $N = 1$ supergravity in eleven dimensions which are only symmetric.

5 An Extension of 11-dimensional Supergravity

As we saw above, one of multiplets to come out of Kostant's equation is the 11-dimensional supergravity one. Since it comes in the form of the $SO(9)$ multiplet, it is natural to consider the light-cone formulation of this model. The dimensionally reduced 4-dimensional $N = 8$ supergravity was constructed in this formulation up to the three-point function some twenty years ago by Bengtsson, Bengtsson and Brink [25]. Recently this action has been "oxidized" to 11 dimensions [26]. This is now a perfect starting point to attempt an extension to a model that contains all the Euler multiplets. The corresponding free such theory was constructed by Brink, Ramond and Xiong [27].

The supergravity theory is described by a superfield in terms of an anticommuting coordinate θ^a which is an $\mathbf{8}$ under $SO(7)$. Following the highest weight representations of the irreps in the multiplet we can write the superfield as

$$\Phi = \theta^1 \theta^8 \, h(y^-, \vec{x}) + \theta^1 \theta^4 \theta^8 \, \psi(y^-, \vec{x}) + \theta^1 \theta^4 \theta^5 \theta^8 \, A(y^-, \vec{x}) + lower \ \ weights, \tag{49}$$

with $y^- = x^- - \frac{i\theta\bar{\theta}}{\sqrt{2}}$. We now like to add internal bosonic degrees of freedom and let Kostant's operator work on the field. We then have to find a representation of the coset. We start by finding an oscillator representation of F_4 In this case, it turns out that all representations of the exceptional group F_4 are generated by three (not four [28]) sets of oscillators transforming as **26**.

We label each copy of 26 oscillators as $A_0^{[\kappa]}$, $A_i^{[\kappa]}$, $i = 1, \cdots, 9$, $B_a^{[\kappa]}$, $a = 1, \cdots, 16$, and their hermitian conjugates, and where $\kappa = 1, 2, 3$. Under $SO(9)$, the $A_i^{[\kappa]}$ transform as **9**, $B_a^{[\kappa]}$ transform as **16**, and $A_0^{[\kappa]}$ is a scalar. They satisfy the commutation relations of ordinary harmonic oscillators

$$[\, A_i^{[\kappa]}, A_j^{[\kappa']\,\dagger}\,] = \delta_{ij}\, \delta^{[\kappa]\,[\kappa']} \, , \qquad [\, A_0^{[\kappa]}, A_0^{[\kappa']\,\dagger}\,] = \delta^{[\kappa\,\kappa']} \, . \tag{50}$$

Note that the $SO(9)$ spinor operators satisfy Bose-like commutation relations

$$[\, B_a^{[\kappa]}, B_b^{[\kappa']\,\dagger}\,] = \delta_{ab}\, \delta^{[\kappa]\,[\kappa']} \, . \tag{51}$$

The generators T_{ij} and T_a

$$T_{ij} = -i \sum_{\kappa=1}^{4} \left\{ \left(A_i^{[\kappa]\dagger} A_j^{[\kappa]} - A_j^{[\kappa]\dagger} A_i^{[\kappa]} \right) + \frac{1}{2}\, B^{[\kappa]\dagger} \gamma_{ij} B^{[\kappa]} \right\} \, ,$$

$$T_a = -\frac{i}{2} \sum_{\kappa=1}^{4} \left\{ (\gamma_i)^{ab} \left(A_i^{[\kappa]\dagger} B_b^{[\kappa]} - B_b^{[\kappa]\dagger} A_i^{[\kappa]} \right) - \sqrt{3} \left(B_a^{[\kappa]\dagger} A_0^{[\kappa]} - A_0^{[\kappa]\dagger} B_a^{[\kappa]} \right) \right\} \, ,$$

satisfy the F_4 algebra,

$$[\, T_{ij}, T_{kl}\,] = -i \left(\delta_{jk} T_{il} + \delta_{il} T_{jk} - \delta_{ik} T_{jl} - \delta_{jl} T_{ik} \right) , \tag{52}$$

$$[\, T_{ij}, T_a\,] = \frac{i}{2}\, (\gamma_{ij})_{ab}\, T_b \, , \tag{53}$$

$$[\, T_a, T_b\,] = \frac{i}{2}\, (\gamma_{ij})_{ab}\, T_{ij} \, , \tag{54}$$

so that the structure constants are given by

$$f_{ij\,ab} = f_{ab\,ij} = \frac{1}{2}\, (\gamma_{ij})_{ab} \, .$$

We can now use the generators T_a in Kostants equation and we can form the γ's from the knowledge about S_{ij}. The γ's will be written in terms of the θ's and their derivatives. It is then straightforward but tedious to solve Kostant's

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eqation and we will obtain an infinite superfield written in terms of ordinary fields depending on x expanded in θ's and creation operators of the oscillator algebra. For details see [27].

The next step to see is if also for this theory one can construct a three-point function along the lines of the corresponding work for the supergravity multiplet. Note that this theory is different from other approaches to find higher spin gauge theories in that it is built from triplets with very strong cancellations between the bosons and the fermions. If this is good enough to warrant a finite quantum theory rests to be seen.

6 Afterthoughts

In this talk I have tried to show how the idea of fermionic degress of freedom introduced via Grassmann odd coordinates has led us up to the Standard Model of particle physics and onto the Superstring Theory and perhaps further on. Some thirty years ago when we discussed fermionic coordinates we were very defensive and said that this is just a book-keeping device, but time has shown that it is more than that. The mathematics behind them is slightly different and we cannot grasp them like the three space dimensions around us. We were also ridiculed when we talked about 26 or 10 dimensions of space-time and also there we were defensive. Now the extra dimensions are taken for granted and in the same way as we have learnt to work with them we have learnt to work with the fermionic ones. I think this gives us a healthy perspective on the concepts of coordinates in general.

What would have been Gunnar Nordström's role had he not dribbled into research on radioactivity and forgotten uranium in his waist pocket? Had he been like a Sandels in Pardala by or a Döbeln at Jutas leading the troups into the new field of quantum physics and new coordinates or more like the brave leutenant Zidén storming along far in front of everybody [29]. Unfortunately we cannot tell. Science lost one of its most inventive researchers when Gunnar Nordström died so young, and the developments of modern physics was slowed down. Even so the development is most impressive.

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